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Soliton interaction in nonequilibrium dynamical systems

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We propose an approach to numerical analysis of soliton interactions in dynamical systems described by the Ginzburg-Landau equation. This approach is based on the analysis of soliton trajectories in a phase plane. The main features of the interaction are represented in sufficient detail to permit understanding of the formation of bound states and their stability. Among the bound states we find some with $\pi/2$ out-of-phase solitons.

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I. INTRODUCTION

The effect of soliton interactions clearly demonstrates the particlelike nature of solitons. For the unperturbed nonlinear Schrödinger (NLS) equation (which is integrable by the inverse scattering transform), soliton interactions have been studied in Refs. [1–6] (see also the review paper [7]) and its features are well understood. In particular, it has been found that the interaction of initially motionless solitons depends on the relative phase between them: in-phase solitons attract each other and form periodical solution, while out-of-phase solitons interact repulsively. For arbitrary phases the result is more complicated; however, Refs. [2,6] present an approximate expression, which allows one to find the amplitudes and velocities of emerging solitons.

At the same time, the integrable NLS equation is an approximation that is rarely fulfilled in practice. In many cases, there are additional effects such as third-order dispersion, higher-order nonlinearity, amplification, and damping. The soliton interaction is a very sensitive phenomenon and is greatly affected by perturbations (see, e.g., [7] and references therein). For instance, it was found that third-order dispersion and spectral filtering reduce the interaction [7–12]. However, despite the great attention paid to the soliton interaction in the presence of perturbations, the problem lacks some general approach and sometimes leads to misunderstanding. For example, some controversy appeared on the reduction of the soliton interaction by spectral filtering [8,9,13,14].

Commonly, the soliton interaction in the presence of perturbations is studied numerically, as the problem is too complicated for analytical consideration. However, comprehensive numerical analysis is also hampered because too many parameters are involved, including parameters of the initial condition and the perturbation itself. In particular, numerical studies are often limited to special cases of in-phase and out-of-phase solitons, i.e., to symmetric and antisymmetric initial conditions. In addition, the exact analytical solution for the perturbed NLS equation is known only for a few particular cases, which makes numerical study even more difficult.

In this paper we propose an approach to the numerical analysis of the perturbed soliton interaction. This approach is based on the analysis of soliton trajectories on the (r,ϕ) plane (phase plane), where $r=r(z)$ is the separation between solitons and $\phi=\phi(z)$ is the relative phase between them. We show that, if the initial values $r(0)$ and $\phi(0)$ are given, dynamics of the soliton interaction can be predicted from the analysis of the special points and separatrices on this plane.

II. BASIC EQUATIONS

As a particular example of the perturbed NLS equation we choose the complex quintic Ginzburg-Landau (GL) equation because of the following reasons. The quintic GL equation admits the stable soliton propagation, that the nonsoliton radiation is suppressed, and that soliton parameters (the amplitude and the width) are uniquely determined by the coefficients of the GL equation. In addition, there is great interest in the soliton interaction in the GL equation due to applications to optical communications and fiber lasers [15–17] and to dynamics of binary fluid convection [18].

We write the GL equation in the following form, used in nonlinear optics:

$$i\frac{\partial\psi}{\partial z}+\frac{D}{2}\frac{\partial^2\psi}{\partial\tau^2}+|\psi|^2\psi=i\delta\psi+i\beta\frac{\partial^2\psi}{\partial\tau^2}+i\epsilon|\psi|^2\psi-i\mu|\psi|^4\psi, \tag{1}$$

where τ is the retarded time, z is the propagation distance, and D determines the sign of dispersion. In terms of nonlinear fiber optics, $D=+1$ corresponds to negative or anomalous dispersion and $D=-1$ corresponds to positive or normal dispersion. The terms on the right-hand side (rhs) of Eq. (1) stand for linear amplification, spectral filtering, nonlinear gain, and saturation of the nonlinear gain, respectively.

Strictly speaking, a general pulse solution of Eq. (1) for an arbitrary set of parameters is not known in analytical form, although several solutions were found that exist if some relation between coefficients is fulfilled (see [19] and references therein). If the coefficients on the rhs of Eq. (1) are small, soliton dynamics can be estimated from the perturbation theory. If we write the solution of the unperturbed NLS equation in the form

$$\psi(\tau,z)=\eta\,\mathrm{sech}[\eta(\tau-\Delta(z))]\exp[i\Phi(\tau,z)], \tag{2}$$

then the stationary soliton amplitude obeys the equation

$$\delta-\frac{1}{3}\beta\eta^2+\frac{2}{3}\epsilon\eta^2-\frac{8}{15}\mu\eta^4=0. \tag{3}$$

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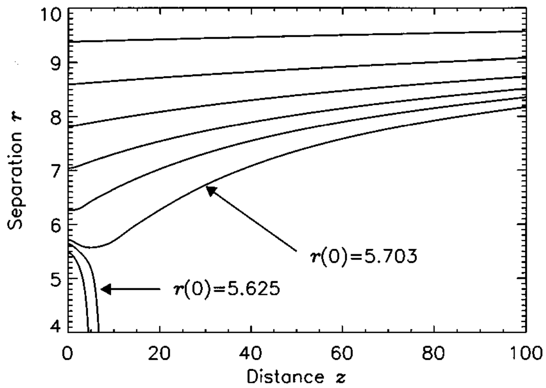

FIG. 1. Soliton separation for the interaction of in-phase solitons at $D=+1$, $\beta=1$, and $k=0.5$ for different Δ. Values of $r(0)$ for two curves near the threshold are shown.

It follows from the analysis in Ref. [8] that the dynamics of the soliton interaction is determined mainly by the value of the spectral filtering β. So it is convenient to choose all the coefficients of the equation to be proportional to β in such a way that $\eta=1$:

$$\delta=-k\beta/3, \quad \epsilon=\beta(1/2+k), \quad \mu=(5/8)k\beta, \tag{4}$$

where the parameter k determines the separation from the special (singular) point $(0,\beta,\beta/2,0)$ in the $(\delta,\beta,\epsilon,\mu)$ parameter space (see [17,19]). We choose $k=0.5$, which is enough to be sure that solitons are far enough from the singularity.

To create initial conditions in the form of superposition of two stationary pulses, we solve Eq. (1) numerically and obtain a stationary one-soliton solution $F(\tau)$ for each set of parameters. Then this stationary solution is used in the initial conditions

$$\psi_0(\tau)=F(\tau-r(0)/2)+\exp(i\phi_0)F(\tau+r(0)/2), \tag{5}$$

where $r(z)$ is the separation between the solitons [the minimum possible variation of $r(0)$ is determined by the numerical grid step size].

III. SOLITON INTERACTION IN THE CASE OF ANOMALOUS DISPERSION

A. In-phase and out-of-phase solitons

Before considering a general behavior, we study the soliton interaction in particular cases of in-phase $(\phi=0)$ and out-of-phase $(\phi=\pi)$ pulses. Let us recall the dynamics of the NLS solitons. If $\phi=0$ and all the coefficients on the rhs of Eq. (1) are equal to zero, initially motionless solitons of the NLS equation attract each other and collide and then repeat this process periodically. In the presence of weak spectral filtering the motion of solitons becomes slower, but the attractive type of interaction is retained at least before the first collision [8]. However, if the spectral filtering is strong enough and solitons are well separated [$r(0)>r_1$, $r_1\approx5.66$ for $\beta=1$], the interaction changes sign from attraction to repulsion. Figure 1 shows the soliton separation versus z for different values of the initial separation r_0. One can see that for $r(0)<r_1$ the solitons still attract each other, while for $r(0)>r_1$ solitons move away from each other until the interaction becomes negligible.

For out-of-phase pulses [$\phi(0)=\pi$], we also observe the change of the sign of the interaction at certain $r(0)$ and an even more remarkable phenomenon: the formation of bound states (BSs) of out-of-phase solitons. In this case, to characterize the interaction, we introduce the critical value $r_2\approx10.82$. For $r(0)<r_2$, the bound state is formed, with the separation between the solitons $r_{\mathrm{BS}}^{(1)}\approx5.4$. Note that, depending on the initial separation, solitons can move toward each other or away from each other during the bound state formation. For $r(0)>r_2$, pulses repel each other as in the case of the unperturbed NLS equation.

B. Arbitrary relative phase

The question arises, what happens if the relative phase between the solitons is neither 0 nor π? This is the most interesting case, although it is more difficult to analyze. Indeed, for $\phi\neq0,\pi$, the soliton interaction is asymmetric, with some oscillations and nonuniform dynamics of both the soliton amplitudes and positions. It is known that in the case of an integrable system [2] (and, more generally, a Hamiltonian system [20]), there is some energy exchange between solitons for $\phi\neq0,\pi$. Consequently, soliton amplitudes become unequal. On the other hand, in our system for $\beta\sim1$ the amplitude and the central frequency are tightened to the stationary values.

To characterize the soliton interaction, we propose to plot the ''soliton interaction trajectory'' on the (r,ϕ) plane, where $r=r(z)$ is the separation between the solitons and $\phi=\phi(z)$ is their relative phase. We suppose $-\pi<\phi\leq\pi$. In numerical simulations we calculate r as the separation between two maxima and ϕ as the phase difference between them. The plots are in the polar frame, i.e., $r\cos(\phi)$ and $r\sin(\phi)$ are plotted as the abscissa and the ordinate, respectively.

Strictly speaking, the soliton shapes can change during the interaction because the dynamical system is infinite dimensional. However, it follows from numerical simulations that these changes are small unless the distance r is smaller than the width of each soliton. So the reduced problem has only two dynamical variables r and ϕ. It is convenient to plot them using the polar coordinates.

First, we analyze the special points on this plane. Note that a small region around the origin is undefined because the distance between the solitons becomes less than the width of each soliton. The dynamics of such strongly overlapping solitons depends on the relative phase between them. If ϕ is close to π and solitons are in the ''forbidden zone'' [filled region in Fig. 2(a)], the amplitude of their superposition is too small and the trivial solution $\psi=0$ is formed. If two strongly overlapping solitons are outside the filled area, they collide and fuse to one pulse.

Each critical value on the line $\phi=0,\pi$ (i.e., r_1 and $r_{\mathrm{BS}}^{(1)}$) is a special point. The circle with the center at the origin of the coordinate frame and radius r_1 can play the role of a separatrix.

Figure 2(a) shows the overall dynamics of the soliton interaction on the (r,ϕ) plane for $D=+1$, $\phi(0)=\pm0.95\pi$,

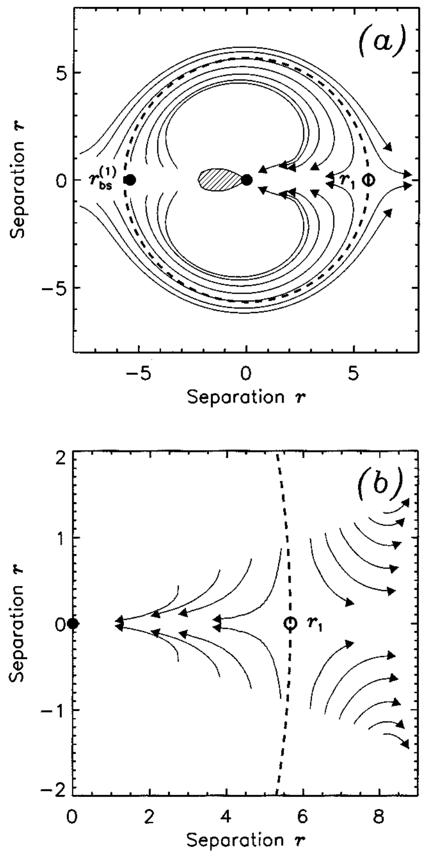

FIG. 2. Soliton trajectories on the (r,ϕ) plane for $D=+1$, $\beta=1$, and $k=0.5$ (solid lines with arrows show the direction of motion). The solid circle shows the center of the frame and the bound state of out-of-phase solitons; the open circle shows the threshold between attraction and repulsion for in-phase solitons. The dashed line gives the circle with radius r_1. The filled area corresponds to the ''forbidden zone.'' (a) $\phi(0)=\pm 0.95\pi$ and (b) $\phi(0)=\pm 0.05\pi$.

and several values of the initial separation $r(0)$ in the vicinity of $r_{\rm BS}^{(1)}$. It can be seen that this is a saddle point in terms of the theory of dynamical systems. Indeed, r tends to the stationary value, while the variation of the phase $|\phi(z)-\phi(0)|$ increases. Note that the separation remains almost constant, until the phase becomes close to zero. The further dynamics is mainly determined by the ratio of $r(0)$ and r_1. If $r(0)<r_1$, the separation starts to decrease, i.e., the solitons attract each other and eventually fuse to a single pulse. If $r(0)>r_1$, the trajectory rounds the circle and tends to the x axis, while the separation increases, i.e., solitons repel each other. One can see that one of the trajectories crosses the circle so that the actual boundary between the regions of attraction and repulsion is determined by a slightly deformed circle. Note that the system under consideration plays the role of a ''phase equalizer'' for the soliton pair in the sense that the phase difference converges to zero, starting from an arbitrary value.

Another case is depicted in Fig. 2(b), which is similar to Fig. 2(a) except for the initial phase $\phi(0)=\pm 0.05\pi$. This can be considered a magnified version of Fig. 2(a). One can clearly see that the circle with radius r_1 is a separatrix for the soliton interaction. The point $\phi=0$ on this curve is again the saddle point. However, stable and unstable directions interchange in comparison with $\phi(0)=0.95\pi$, i.e., the phase tends to zero while the separation increases or decreases, depending on the position of the initial point with respect to the separatrix.

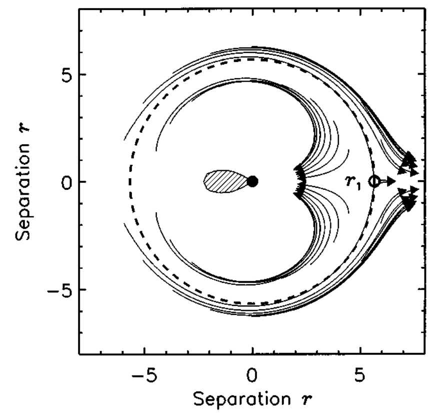

FIG. 3. Dynamics of the soliton interaction for $D=+1$; $\phi(0)=\pm 0.1\pi, \pm 0.2\pi, \ldots, \pm 0.9\pi$; and $r(0)=4.688$ and 6.25.

The same dynamics is observed for other values of the initial phase difference (see Fig. 3). This figure shows soliton trajectories for two values of $r(0)$ and $\phi(0)=0.1\pi,\ 0.2\pi,\ \ldots,\ 0.9\pi$. Soliton trajectories that start outside the circle with radius r_1 round it. Eventually a pair of in-phase solitons is formed, while the separation between solitons increases. Note the trajectory for $\phi(0)=0.9\pi$, which almost coincides with the circle. Another set of trajectories, which start inside the circle, corresponds to fusing solitons. This plot clearly shows that the special point $(r_1,0)$ is a saddle point.

IV. THE CASE OF NORMAL DISPERSION

A. In-phase and out-of-phase solitons

By analogy, we start from the special cases of in-phase and out-of-phase pulses. As for anomalous dispersion, we found numerically the stationary solution for each set of parameters and use the linear superposition (5) of two pulses as initial conditions. For the case of normal dispersion, the adiabatic perturbation theory cannot be applied, but we still use relations (4) between the coefficients on the rhs of Eq. (1) for convenience. It turns out that for $\beta\sim 1$ the amplitude of the stationary pulse in normal dispersion is approximately the same as for anomalous dispersion, but the pulse width is nearly twice as large.

For the case of positive dispersion, the bound states were found for both the in-phase and out-of-phase solitons. For in-phase solitons, numerical study shows that the pulses fuse to a single pulse for $r(0)<r_3$, $r_3\approx 7.7$, form a bound state for $r_3<r(0)<r_4$ (a separation in the bound state is $r_{\rm BS}^{(2)}\approx 8.56$), and repel each other for $r(0)>r_4$, $r_4\approx 11.84$. Note that the pulses in such bound state are much farther apart from each other than in the bound state of out-of-phase solitons in the case of anomalous dispersion.

Out-of-phase pulses form the bound state for $r(0)<r_5$, where $r_5=9.41$, and the separation between the pulses in the bound state $r_{\rm BS}^{(3)}=4.8$. For $r(0)<2.15$ pulses annihilate and a

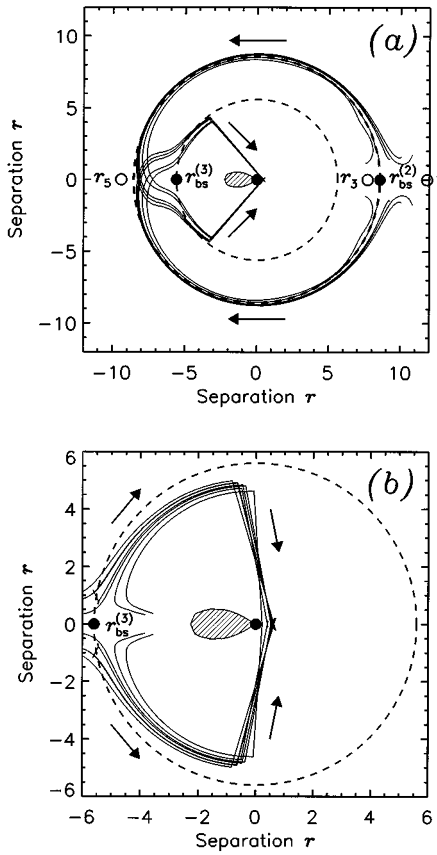

FIG. 4. Soliton trajectories on the (r,ϕ) plane for $D=-1$. (a) $\phi(0)=\pm 0.05\pi$ and (b) $\phi(0)=\pm 0.95\pi$.

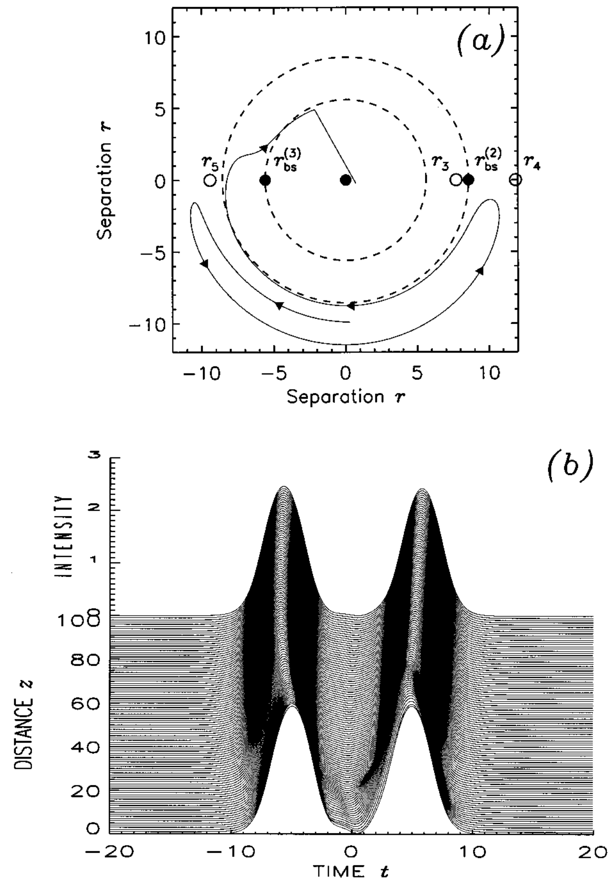

FIG. 5. Dynamics of the $\pi/2$ out-of-phase solitons for $D=-1$, $r(0)=9.922$, and $\phi(0)=-\pi/2$. (a) Phase plane and (b) time domain.

trivial solution $\psi=0$ is formed, while for $r(0)>r_5$ pulses repel each other.

B. Arbitrary phase

Figure 4(a) shows the soliton interaction on the phase plane for $\phi(0)=0.05\pi$. The tendency of the phase dynamics is opposite the case shown in Fig. 2: the absolute value of the relative phase increases and passes the point where $\phi=\pm\pi$. Before this point, all trajectories demonstrate a clear tendency to move near the $r_{\mathrm{BS}}^{(2)}$ circle. Hence the special point $r_{\mathrm{BS}}^{(2)}$ is also a saddle point. After reaching the π phase difference, solitons are attracted by the special point $r_{\mathrm{BS}}^{(3)}$ and they continue their motion along the $r_{\mathrm{BS}}^{(3)}$ circle. Finally, when the phase difference $|\phi|\approx 3\pi/4$, the trajectories fall to the center of coordinates, i.e., solitons fuse to a single pulse. We have to emphasize that the value of the phase difference, when the fusion occurs, is almost independent of $r(0)$. This fact cannot be established without use of the (r,ϕ) plane. However, the fusion occurs at different distances, depending on the initial soliton separation.

For $\phi(0)=0.95\pi$, we observe very similar dynamics [see Fig. 4(b)]. In this case the absolute value of the soliton phase *decreases*. There is no contradiction with the case shown in Fig. 4(a), as increasing of the absolute phase occurs along the $r_{\mathrm{BS}}^{(2)}$ circle, while decreasing takes the plane along the $r_{\mathrm{BS}}^{(3)}$ circle. For $|\phi|\approx\pi/2$, solitons fuse. Again, the value of the phase difference at which the solitons fuse is almost independent on the initial separation, although the distance where this occurs strongly depends on $r(0)$ [it increases as $r(0)$ increases].

Finally, using our method, we discover one more bound state for $\phi(0)=\pi/2$ and $r(0)\approx 9.922$. When the initial condition is close to the point corresponding to this bound state, both the soliton separation and the relative phase between pulses oscillate [Fig. 5(a)]. The closer the initial condition to this stationary point, the more oscillations can be observed. In other words, the trajectories are outgoing spirals, so that the special point inside them is the unstable focus. The dynamics of the soliton interaction in time domain is shown in Fig. 5(b). Note the asymmetry of the field between the two pulses.

V. DISCUSSION AND CONCLUSION

As we can see from the results obtained, all stationary points on the $\phi=0,\pi$ line are of saddle type. If the BSs on this line is stable to perturbations of r, it is unstable to phase perturbations and vice versa. We can compare these results with the results obtained in [20] for Hamiltonian systems, where soliton BSs were also found to be unstable.

At the same time, although all the bound states studied appear to be unstable, they clearly manifest themselves in the interaction. If we choose initial conditions close to the special point, solitons propagate a long distance before eventually fusing or moving apart.

These results allows the understanding of an earlier paper [21], where the BS of in-phase solitons was discovered. The authors of that paper claimed that the BS is stable in a small asymmetric perturbation, although it decays if the perturbation is strong enough. Apparently, their simulations were similar to those shown in Fig. 3(a). After being slightly perturbed, the BS propagates a large distance with almost constant separation between solitons; however, eventually the solitons fuse to one.

In conclusion, we study the soliton interaction of the quintic Ginzburg-Landau equation. We show that the dynamics of the soliton interaction is mainly determined by the position of initial conditions with respect to the special points and the separatrix on the (r,ϕ) plane. The special points correspond to bound states of two solitons. All the bound states studied appear to be unstable with respect to either the phase or separation perturbations. Despite this, bound states play a pivotal role in the overall dynamics of two-soliton interactions.

ACKNOWLEDGMENT

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